

BFKL Azimuthal Imprints in Inclusive Three-jet Production at 7 and 13 TeV

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Abstract

We propose the study of new observables in LHC inclusive events with three tagged jets, one in the forward direction, one in the backward direction and both well-separated in rapidity from the each other (Mueller-Navelet jets), together with a third jet tagged in central regions of rapidity. Since non-tagged associated mini-jet multiplicity is allowed, we argue that projecting the cross sections on azimuthal-angle components can provide several distinct tests of the BFKL dynamics. Realistic LHC kinematical cuts are introduced.

1 Introduction

In recent years the steady running of the Large Hadron Collider (LHC) has opened up new avenues for the study of the high energy limit of Quantum Chromodynamics (QCD). This is particularly important in the context of jet production since the abundance of data allows for the possibility to study more exclusive observables, needed to isolate regions of phase hidden in more inclusive, previous, analysis. In this work we focus on the investigation of jet production in the so-called multi-Regge kinematics. When jets are produced at large relative rapidities the Balitsky-Fadin-Kuraev-Lipatov

(BFKL) approach in the leading logarithmic (LL) [1, 2, 3, 4, 5, 6] and next-to-leading logarithmic (NLL) approximation [7, 8] offers an effective framework to calculate the bulk of the cross sections.

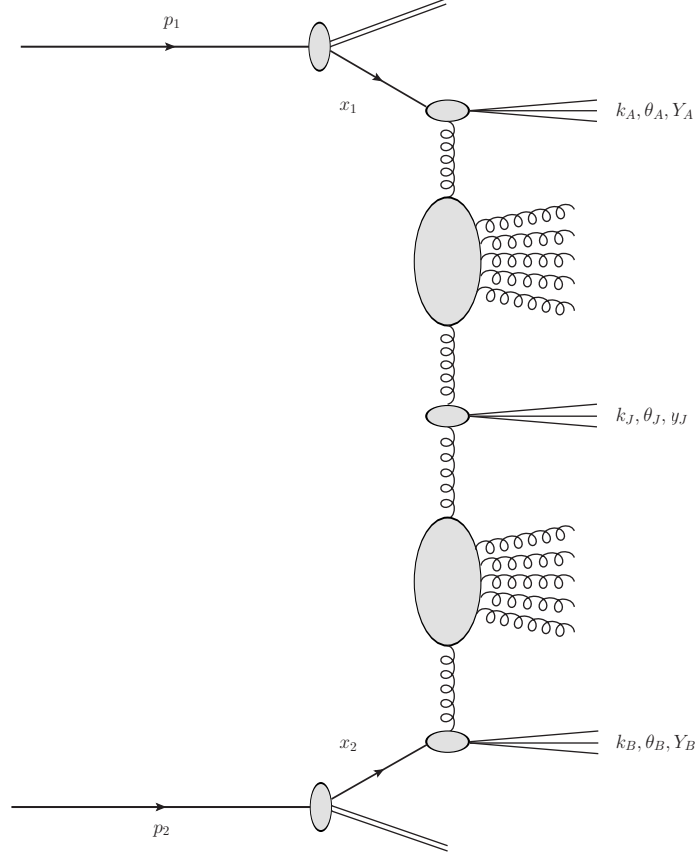


Figure 1: Inclusive three-jet production process in multi-Regge kinematics.

Mueller-Navelet jets [9] correspond to the inclusive hadroproduction of two jets¹ with large and similar transverse momenta, $k_{A,B}$, and a significant relative separation in rapidity $Y = \ln(x_1 x_2 s / (k_A k_B))$, where $x_{1,2}$ are the longitudinal momentum fractions of the partons generating the jets and s is the centre-of-mass-energy squared s . Different studies [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] of the average values, $\langle \cos(m\phi) \rangle$, for the azimuthal-angle formed by the two tagged jets, ϕ , have shown the presence of a large soft gluon activity populating the rapidity gap. These observables are, however, strongly affected by collinear effects [24, 25], stemming from

¹ Another interesting idea, suggested in [10] and investigated in [11], is the study of the production of two charged light hadrons, π^\pm , K^\pm , p , \bar{p} , with large transverse momenta and well separated in rapidity.

the $n = 0$ Fourier component in ϕ of the BFKL kernel. This dependence is removed if instead the ratios of projections on azimuthal-angle observables $\mathcal{R}_n^m = \langle \cos(m\phi) \rangle / \langle \cos(n\phi) \rangle$ [24, 25] (where m, n are integers and ϕ the azimuthal angle between the two tagged jets) are introduced. In particular, these also offer a more clear signal of BFKL effects than the standard predictions for the growth of hadron structure functions $F_{2,L}$ (well fitted within NLL approaches [26, 27]). The comparison of different NLL predictions for these ratios \mathcal{R}_n^m [28, 29, 30, 31, 32] with LHC experimental data has been very successful.

We understand the current situation as the beginning of precision physics using the BFKL formalism. Within the framework itself there exist many theoretical questions to be answered. Some of these include to find out what is the more accurate way to implement the running of the coupling, if there is any onset of saturation effects at the level of exclusive observables, how to isolate BFKL dynamics from multiple interaction effects, etc. To address these issues it is important to investigate even more exclusive final states.

Here we advance in this direction by proposing new observables associated to the inclusive production of three jets: two of them are the original Mueller-Navelet jets and the third one is a tagged jet in central regions of rapidity (see Fig. 1). Experimentally, they have the advantage to belong to the already recorded Mueller-Navelet events, it only requires of further binning in the internal jets. Theoretically, they will allow us to better understand distinct features of the BFKL ladder, in other words, to find out which ones of its predictions cannot be reproduced by other approaches such as low order exact perturbation theory or general-purpose Monte Carlo event generators. Parton-level studies have been recently presented in [33] while here we focus on calculating realistic cross-sections at the LHC.

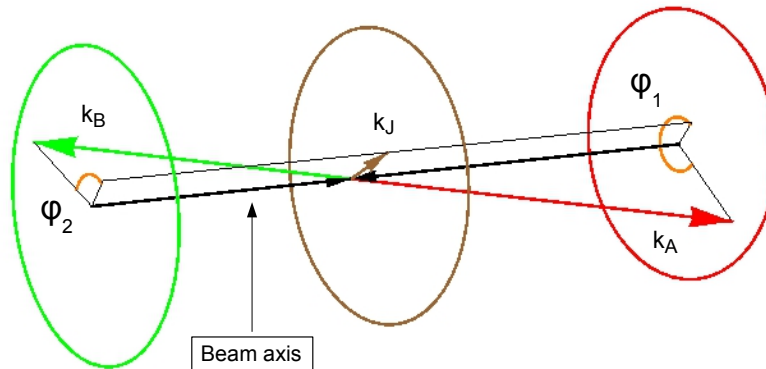


Figure 2: Representation of a three-jet event in a generic detector. All three circles are perpendicular to the beam axis.

In order to focus our discussion, we will present results for the above mentioned \mathcal{R}_n^m ratios but now with a further dependence on the p_t and rapidity of the central jet. As a novel result, we will also present predictions for the new ratios

$$\mathcal{R}_{PQ}^{MN} = \frac{\langle \cos(M\phi_1) \cos(N\phi_2) \rangle}{\langle \cos(P\phi_1) \cos(Q\phi_2) \rangle}, \quad (1)$$

where ϕ_1 and ϕ_2 are, respectively, the azimuthal angle difference between the first and the second (central) jet and between this one and the third jet (see Fig. 2).

A further natural development in this direction has been the extension of these observables to the case of four-jet production in multi-Regge kinematics with a second tagged jet being produced in the central region of rapidity [34, 35]. This allows for the study of even more differential distributions in the transverse momenta, azimuthal angles and rapidities of the two central jets, for fixed values of the four momenta of the two forward (originally Mueller-Navelet) jets. The main observable \mathcal{R}_{PQR}^{MNL} proposed at parton level in [34] is the extension of the one in Eq. (1), using three cosines instead of two in numerator and denominator. This observable also paves the way for detailed studies of multiple parton scattering [36, 37, 38, 39, 40] although we should point out that in Ref. [41] there is a claim that multiple parton interactions (MPI) are negligible in the present LHC kinematics for the values of transverse momenta used in the following.

In the next two Sections, we focus on the case of inclusive three-jet production performing a realistic study beyond the parton level calculation.

This will allow for a comparison of our observables with forthcoming analysis of the LHC experimental data. Cross-sections are calculated using collinear factorization to produce the two most forward/backward jets, convoluting the “hard” differential cross section, which follows the BFKL dynamics, with collinear parton distribution functions included in the forward “jet vertex” [42, 43, 44, 45, 46, 47, 48]. We link these two Mueller-Navelet jet-vertices with the centrally produced jet via two BFKL gluon Green functions. To simplify our predictions, we integrate over the momenta of all produced jets, using current LHC experimental cuts, only fixing the rapidity of the central jet to lie in the middle of the two most forward/backward tagged jets. In the following Section we will show the main formulas, in the next-to-last Section we will present our numerical predictions to finally end with our Summary and Outlook.

2 Hadronic inclusive three-jet production in multi-Regge kinematics

The process under investigation (see Figs. 1, 2 and 3) is the production of two forward/backward jets, both characterized by high transverse momenta $\vec{k}_{A,B}$ and well separated in rapidity, together with a third jet produced in the central rapidity region and with possible associated mini-jet production. This corresponds to

$$\text{proton}(p_1) + \text{proton}(p_2) \rightarrow \text{jet}(k_A) + \text{jet}(k_J) + \text{jet}(k_B) + \text{minijets} . \quad (2)$$

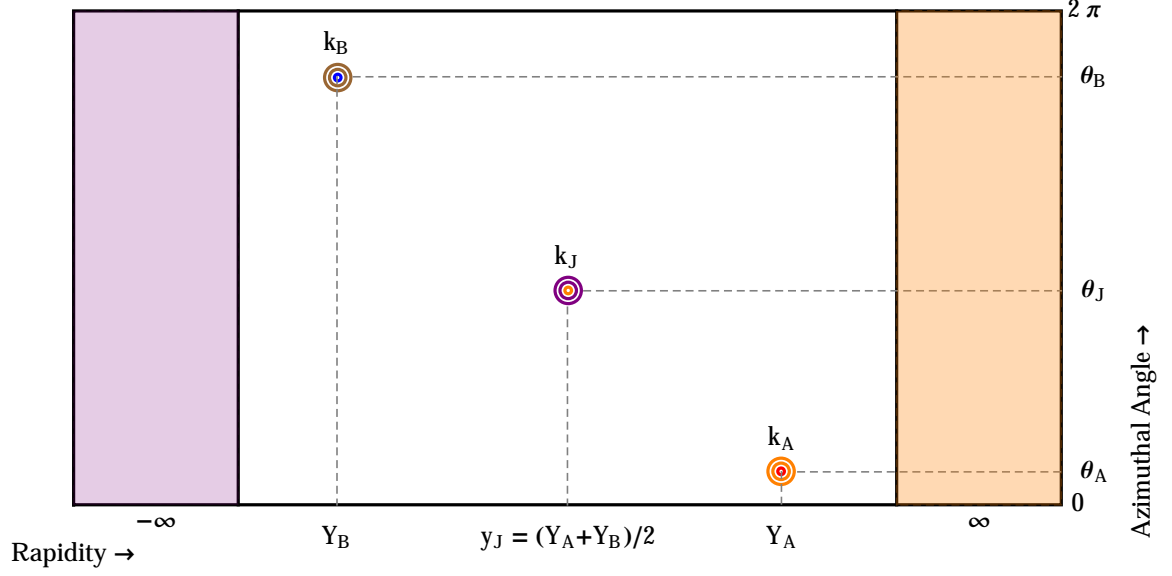


Figure 3: A primitive lego plot depicting a three jet event. k_A is a forward jet with large positive rapidity Y_A and azimuthal angle θ_A , k_B is a forward jet with large negative rapidity Y_B and azimuthal angle θ_B and k_J is a central jet with rapidity y_J and azimuthal angle θ_J .

In collinear factorization the cross section for the process (2) reads

$$\frac{d\sigma^{3\text{-jet}}}{dk_A dY_A d\theta_A dk_B dY_B d\theta_B dk_J dy_J d\theta_J} = \sum_{r,s=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 f_r(x_1, \mu_F) f_s(x_2, \mu_F) d\hat{\sigma}_{r,s}(\hat{s}, \mu_F) , \quad (3)$$

where the r, s indices specify the parton types (quarks $q = u, d, s, c, b$; anti-quarks $\bar{q} = \bar{u}, \bar{d}, \bar{s}, \bar{c}, \bar{b}$; or gluon g), $f_{r,s}(x, \mu_F)$ are the initial proton PDFs; $x_{1,2}$ represent the longitudinal fractions of the partons involved in the hard subprocess; $d\hat{\sigma}_{r,s}(\hat{s}, \mu_F)$ is the partonic cross section for the production of jets and $\hat{s} \equiv x_1 x_2 s$ is the squared center-of-mass energy of the hard subprocess (see Fig. 1). The BFKL dynamics enters in the cross-section for the partonic hard subprocess $d\hat{\sigma}_{r,s}$ in the form of two forward gluon Green functions φ to be described below.

Using the definition of the jet vertex in the leading order approxima-

tion [42], we can present the cross section for the process as

$$\begin{aligned}
& \frac{d\sigma^{3\text{-jet}}}{dk_A dY_A d\theta_A dk_B dY_B d\theta_B dk_J dy_J d\theta_J} = \\
& \frac{8\pi^3 C_F \bar{\alpha}_s^3}{N_C^3} \frac{x_{J_A} x_{J_B}}{k_A k_B k_J} \int d^2 \vec{p}_A \int d^2 \vec{p}_B \delta^{(2)}(\vec{p}_A + \vec{k}_J - \vec{p}_B) \\
& \times \left(\frac{N_C}{C_F} f_g(x_{J_A}, \mu_F) + \sum_{r=q, \bar{q}} f_r(x_{J_A}, \mu_F) \right) \\
& \times \left(\frac{N_C}{C_F} f_g(x_{J_B}, \mu_F) + \sum_{s=q, \bar{q}} f_s(x_{J_B}, \mu_F) \right) \\
& \times \varphi(\vec{k}_A, \vec{p}_A, Y_A - y_J) \varphi(\vec{p}_B, \vec{k}_B, y_J - Y_B). \quad (4)
\end{aligned}$$

In order to lie within multi-Regge kinematics, we have considered the ordering in the rapidity of the produced particles $Y_A > y_J > Y_B$, while k_J^2 is always above the experimental resolution scale. $x_{J_{A,B}}$ are the longitudinal momentum fractions of the two external jets, linked to the respective rapidities $Y_{J_{A,B}}$ by the relation $x_{J_{A,B}} = k_{A,B} e^{\pm Y_{J_{A,B}}/\sqrt{s}}$. φ are BFKL gluon Green functions normalized to $\varphi(\vec{p}, \vec{q}, 0) = \delta^{(2)}(\vec{p} - \vec{q})$ and $\bar{\alpha}_s = N_c/\pi \alpha_s(\mu_R)$.

Building up on the work in Ref. [33, 34], we study observables for which the BFKL approach will be distinct from other formalisms and also rather insensitive to possible higher order corrections. We focus on new quantities whose associated distributions are different from the ones which characterize the Mueller-Navelet case, though still related to the azimuthal-angle correlations by projecting the differential cross section on the two relative azimuthal angles between each external jet and the central one $\Delta\theta_{AJ} = \theta_A - \theta_J - \pi$ and $\Delta\theta_{JB} = \theta_J - \theta_B - \pi$ (see Fig. 3). Taking into account the factors coming from the jet vertices, it is possible to rewrite Eq. (7) of [33] in the form

$$\begin{aligned}
& \frac{\int_0^{2\pi} d\theta_A \int_0^{2\pi} d\theta_B \int_0^{2\pi} d\theta_J \cos(M\Delta\theta_{AJ}) \cos(N\Delta\theta_{JB})}{d\sigma^{3\text{-jet}}} = \\
& \frac{8\pi^4 C_F \bar{\alpha}_s^3}{N_C^3} \frac{x_{J_A} x_{J_B}}{k_A k_B} \left(\frac{N_C}{C_F} f_g(x_{J_A}, \mu_F) + \sum_{r=q, \bar{q}} f_r(x_{J_A}, \mu_F) \right) \\
& \times \left(\frac{N_C}{C_F} f_g(x_{J_B}, \mu_F) + \sum_{s=q, \bar{q}} f_s(x_{J_B}, \mu_F) \right) \sum_{L=0}^N \binom{N}{L} (k_J^2)^{\frac{L-1}{2}} \\
& \times \int_0^\infty dp^2 (p^2)^{\frac{N-L}{2}} \int_0^{2\pi} d\theta \frac{(-1)^{M+N} \cos(M\theta) \cos((N-L)\theta)}{\sqrt{(p^2 + k_J^2 + 2pk_J \cos\theta)^N}} \\
& \times \varphi_M(k_A^2, p^2, Y_A - y_J) \varphi_N(p^2 + k_J^2 + 2pk_J \cos\theta, k_B^2, y_J - Y_B), \quad (5)
\end{aligned}$$

where

$$\varphi_n(k^2, q^2, y) = 2 \int_0^\infty d\nu \cos\left(\nu \ln \frac{k^2}{q^2}\right) \frac{e^{\bar{\alpha}_s \chi_{|n|}(\nu)y}}{\pi \sqrt{k^2 q^2}}, \quad (6)$$

$$\chi_n(\nu) = 2\psi(1) - \psi\left(\frac{1+n}{2} + i\nu\right) - \psi\left(\frac{1+n}{2} - i\nu\right) \quad (7)$$

(ψ is the logarithmic derivative of Euler's gamma function).

The related experimental observable we propose corresponds to the mean value (with M, N being positive integers)

$$\begin{aligned} \mathcal{C}_{MN} &= \langle \cos(M(\theta_A - \theta_J - \pi)) \cos(N(\theta_J - \theta_B - \pi)) \rangle \\ &= \frac{\int_0^{2\pi} d\theta_A d\theta_B d\theta_J \cos(M(\theta_A - \theta_J - \pi)) \cos(N(\theta_J - \theta_B - \pi)) d\sigma^{3\text{-jet}}}{\int_0^{2\pi} d\theta_A d\theta_B d\theta_J d\sigma^{3\text{-jet}}}. \end{aligned} \quad (8)$$

From a phenomenological point of view, since our main target is to provide testable predictions compatible with the current and future experimental data, we now introduce those kinematical cuts already in place at the LHC. For this purpose, we integrate $\mathcal{C}_{M,N}$ over the momenta of the tagged jets in the form

$$\begin{aligned} C_{MN} &= \\ &\int_{Y_A^{\min}}^{Y_A^{\max}} dY_A \int_{Y_B^{\min}}^{Y_B^{\max}} dY_B \int_{k_A^{\min}}^{k_A^{\max}} dk_A \int_{k_B^{\min}}^{k_B^{\max}} dk_B \int_{k_J^{\min}}^{k_J^{\max}} dk_J \delta(Y_A - Y_B - Y) \mathcal{C}_{MN}, \end{aligned} \quad (9)$$

where the forward/backward jet rapidities are taken in the range delimited by $Y_A^{\min} = Y_B^{\min} = -4.7$ and $Y_A^{\max} = Y_B^{\max} = 4.7$, keeping their difference $Y \equiv Y_A - Y_B$ fixed at definite values in the range $5 < Y < 9$.

From a more theoretical perspective, it is important to have as good as possible perturbative stability in our predictions (see [17] for a related discussion). This can be achieved by removing the contribution stemming from the zero conformal spin, which corresponds to the index $n = 0$ in Eq. (6). We, therefore, introduce the ratios

$$R_{PQ}^{MN} = \frac{C_{MN}}{C_{PQ}} \quad (10)$$

which are free from any $n = 0$ dependence. We proceed now to present our numerical results for a number of different kinematic configurations.

3 Numerical results for azimuthal-angle dependences

We now study the ratios $R_{PQ}^{MN}(Y)$ in Eq. (10) as functions of the rapidity difference Y between the most forward and the most backward jets for a set

of characteristic values of M, N, P, Q and for two different center-of-mass energies: $\sqrt{s} = 7$ and $\sqrt{s} = 13$ TeV. Since we are integrating over k_A and k_B , we have the opportunity to impose either symmetric or asymmetric cuts, as it has been previously done in the Mueller-Navelet case [21, 31]. To be more precise, we study the two kinematical configurations:

1. $k_A^{\min} = 35$ GeV, $k_B^{\min} = 35$ GeV, $k_A^{\max} = k_B^{\max} = 60$ GeV (symmetric);
2. $k_A^{\min} = 35$ GeV, $k_B^{\min} = 50$ GeV, $k_A^{\max} = k_B^{\max} = 60$ GeV (asymmetric).

In order to be as close as possible to the rapidity ordering characteristic of multi-Regge kinematics, we set the value of the central jet rapidity such that it is equidistant to Y_A and Y_B by imposing the condition $y_J = \frac{Y_A + Y_B}{2}$. Moreover, since by tagging a central jet we are able to extract more exclusive information from our observables, we allow three possibilities for the transverse momentum k_J , that is, $20 \text{ GeV} < k_J < 35 \text{ GeV}$ (bin-1), $35 \text{ GeV} < k_J < 60 \text{ GeV}$ (bin-2) and $60 \text{ GeV} < k_J < 120 \text{ GeV}$ (bin-3). Keeping in mind that the forward/backward jets have transverse momenta in the range $[35 \text{ GeV}, 60 \text{ GeV}]$, restricting the value of k_J within these three bins allows us to see how the ratio $R_{PQ}^{MN}(Y)$ changes behaviour depending on the relative size of the central jet when compared to the forward/backward ones. Bin-1, bin-2 and bin-3 correspond to k_J being smaller than, similar to and larger than k_A, k_B , respectively.

Before we proceed to present our numerical results, we should note that we performed the numerical computation of the ratios \mathcal{R}_{PQ}^{MN} both in FORTRAN and in MATHEMATICA (mainly for cross-checks). The NLO MSTW 2008 PDF sets [49] were used and for the strong coupling α_s we chose a two-loop running coupling setup with $\alpha_s(M_Z) = 0.11707$. We made extensive use of the integration routine *Vegas* [50] as implemented in the *Cuba* library [51, 52]. Furthermore, we used the *Quadpack* library [53] and a slightly modified version of the *Psi* [54] routine.

In the following, we present our results collectively in four figures. In Figs. 4 and 6 different ratios are shown for $\sqrt{s} = 7$ TeV and in Figs. 5 and 7 we see the same ratios for $\sqrt{s} = 13$ TeV. In all four figures, in the left column we place the plots for the symmetric kinematic cut ($k_B^{\min} = 35$ GeV) and in the right column the plots for the asymmetric one ($k_B^{\min} = 50$ GeV). The red dot-dashed curve corresponds to k_J bounded in bin-1, the green dashed curve to k_J bounded in bin-2 and finally the blue continuous one to k_J bounded in bin-3. In total, we show the results for six different observables: R_{22}^{11} , R_{12}^{13} , R_{12}^{22} , R_{12}^{23} , R_{12}^{33} and R_{22}^{33} .

The first observation that becomes apparent from a preliminary view to the four figures is that the dependence of the different observables on the rapidity difference between k_A and k_B is rather smooth. This is more pronounced when we consider k_J being larger than the forward/backward jets (blue line). Indeed, the blue curve, which corresponds to large values of

the transverse momentum in the central jet, is mostly linear. The other two curves (red and green) follow generally the same smooth with Y behavior although they tend to be less linear than the blue curve.

The slope of the three curves, in absolute values, depends on the particular observable. For example, in Fig. 4, the blue curve in the top left drops from ~ 3.5 at $Y = 5$ to ~ 4.5 at $Y = 9$, whereas in bottom left it drops from ~ 0.7 to ~ 0.6 .

Another interesting observation is that there are ratios for which changing from the symmetric to the asymmetric cut makes no real difference and other ratios for which the picture changes radically. A characteristic example of the former case is the observable R_{12}^{22} in Fig. 4 bottom line, where we see practically no big differences between the left and right plots. If instead we focus on R_{12}^{13} in Fig. 4 middle line, we see that going from the symmetric cut (left) to the asymmetric one (right) brings forward a big change.

The main conclusion we would like to draw after comparing Fig. 4 to Fig. 5 and Fig. 6 to Fig. 7 is that, in general, for most of the observables there are no significant changes when we increase the colliding energy from 7 to 13 TeV. This is indeed remarkable since it indicates that a sort of asymptotic regime has been reached for the kinematical configurations included in our analysis. It also tells us that our observables are really as insensitive as possible to effects which have their origin outside the BFKL dynamics and which normally cannot be isolated (*e.g.* influence from the PDFs).

4 Summary & Outlook

We have presented a first full phenomenological study of inclusive three-jet production at the LHC within the BFKL framework, focussing on the study of azimuthal-angle dependent observables. Following the work done in Ref. [33] where a new family of observables was proposed to probe the window of applicability of BFKL at the LHC, we have studied here a selection of these observables at a hadronic level (with PDFs) at two different colliding energies, $\sqrt{s} = 7, 13$ TeV. We have considered a symmetric and an asymmetric kinematic cut with respect to the transverse momentum of the forward (k_A) and backward (k_B) jets. In addition, we have chosen to impose an extra condition on the value of the transverse momentum k_J of the central jet, dividing the allowed region for k_J into three sub-regions: k_J smaller than $k_{A,B}$, k_J similar to $k_{A,B}$ and k_J larger than $k_{A,B}$.

We have shown how our observables R_{22}^{11} , R_{12}^{13} , R_{12}^{22} , R_{12}^{23} , R_{12}^{33} and R_{22}^{33} change when we vary the rapidity difference Y between k_A and k_B from 5 to 9 units. We notice a generally smooth functional dependence of the ratios on Y . These observables do not considerably change when we increase the colliding energy from 7 to 13 TeV which assures us that they capture the essence of what the BFKL dynamics dictates regarding the az-

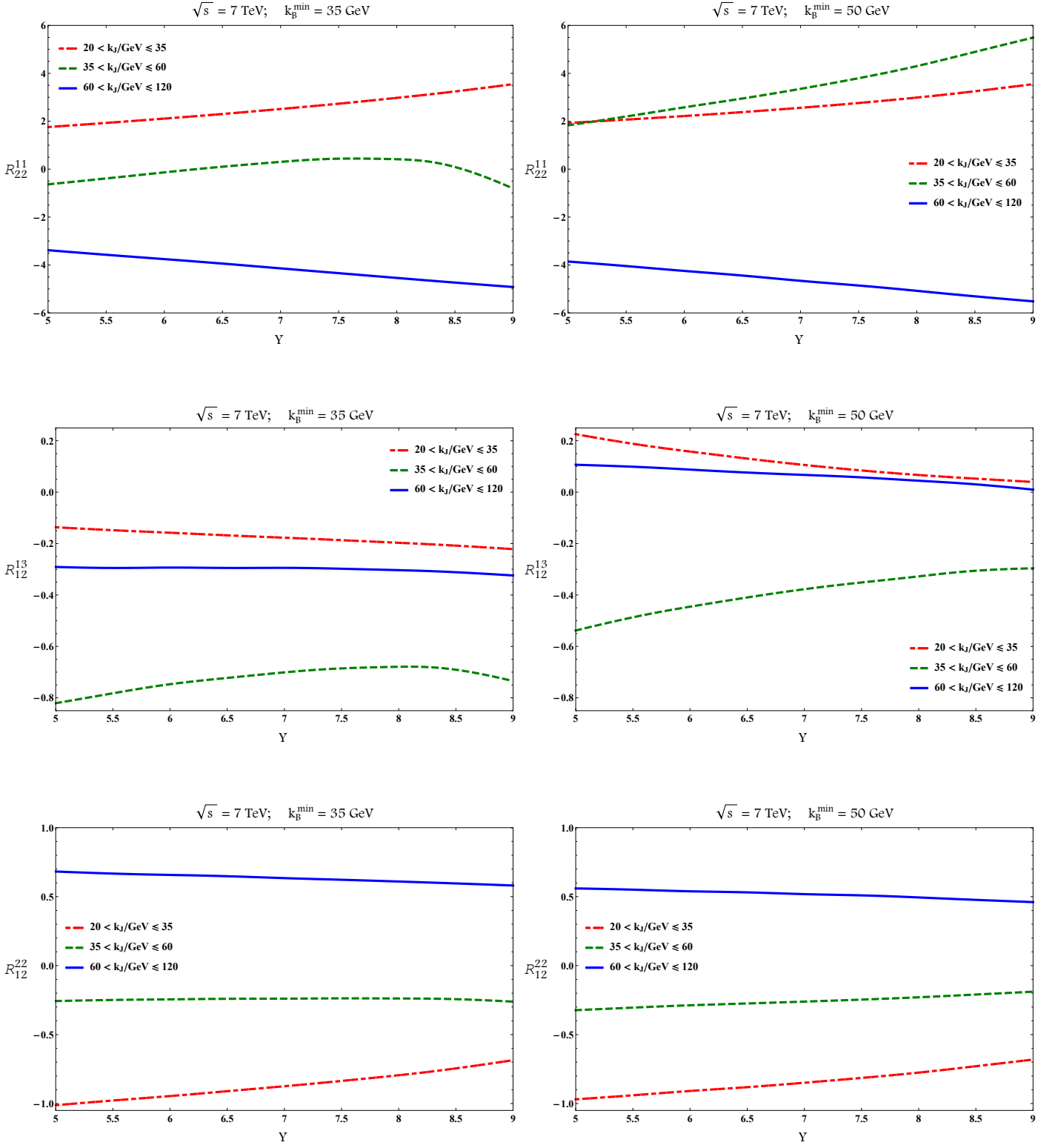


Figure 4: Y -dependence of R_{22}^{11} , R_{12}^{13} and R_{12}^{22} for $\sqrt{s} = 7$ TeV and $k_B^{\min} = 35$ GeV (left column) and $k_B^{\min} = 50$ GeV (right column).

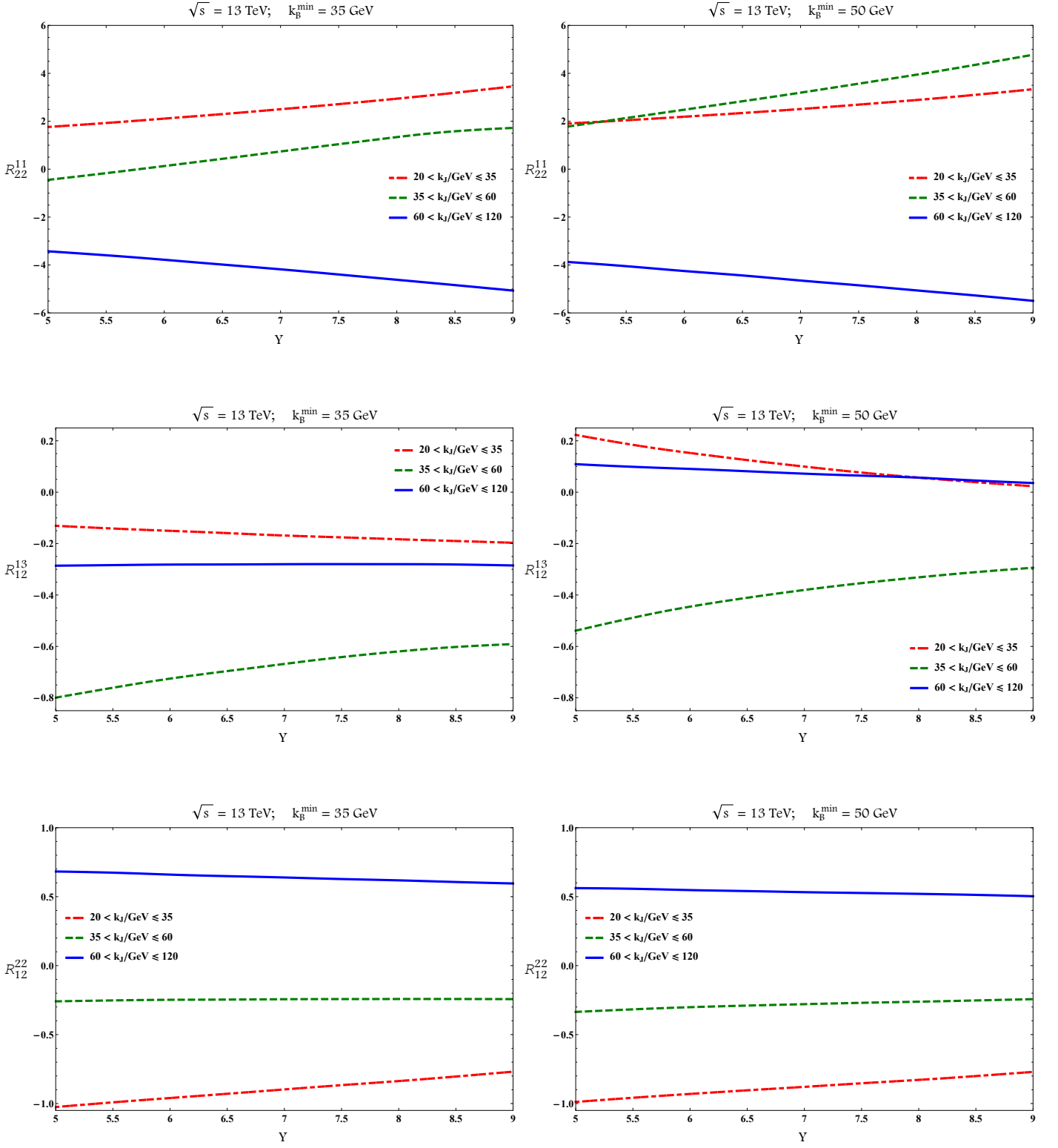


Figure 5: Y -dependence of R_{22}^{11} , R_{12}^{13} and R_{12}^{22} for $\sqrt{s} = 13$ TeV and $k_B^{\min} = 35$ GeV (left column) and $k_B^{\min} = 50$ GeV (right column).

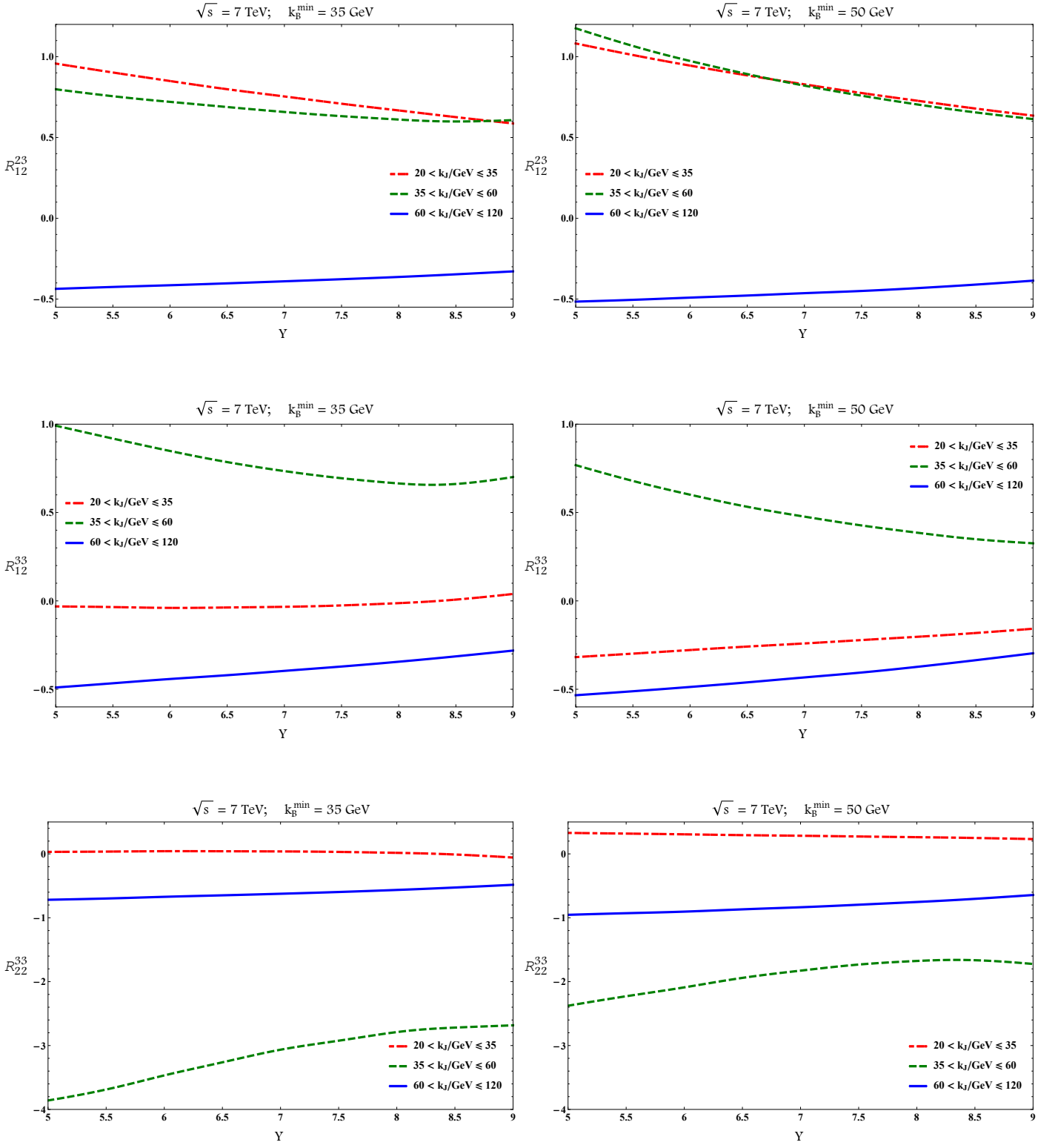


Figure 6: Y -dependence of R_{12}^{23} , R_{12}^{33} and R_{22}^{33} for $\sqrt{s} = 7$ TeV and $k_B^{\min} = 35$ GeV (left column) and $k_B^{\min} = 50$ GeV (right column).

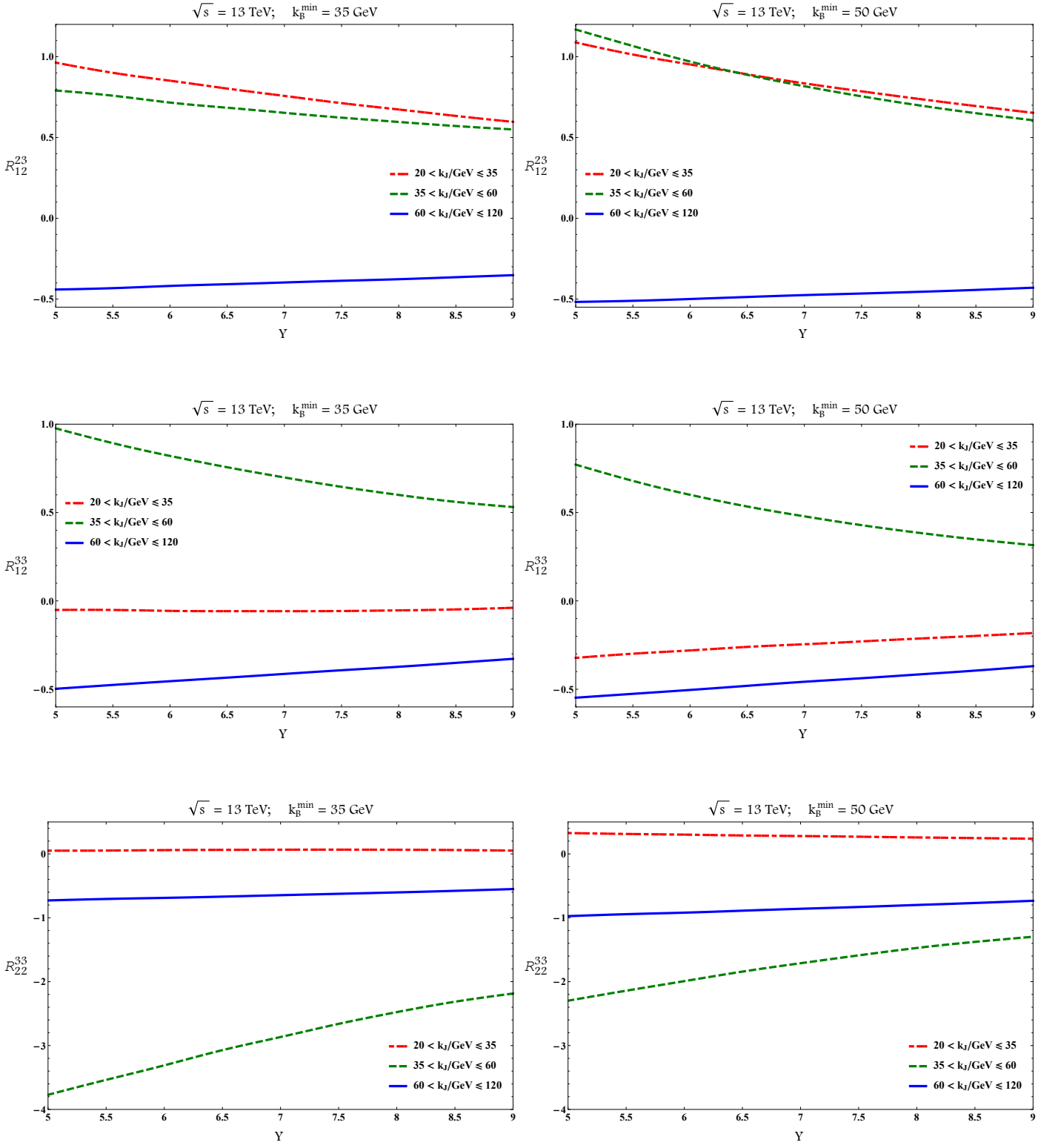


Figure 7: Y -dependence of R_{12}^{23} , R_{12}^{33} and R_{22}^{33} for $\sqrt{s} = 13$ TeV and $k_B^{\min} = 35$ GeV (left column) and $k_B^{\min} = 50$ GeV (right column).

imutal behavior of the hard jets in inclusive three-jet production. It will be very interesting to compare with possible predictions for these observables from fixed order analyses as well as from the BFKL inspired Monte Carlo BFKLex [55, 56, 57, 58, 59, 60, 61]. Predictions from general-purpose Monte Carlos should also be put forward.

Most importantly though, it would be extremely interesting to see an experimental analysis for these observables using the existing and future LHC data. We would like to motivate our experimental colleagues to proceed to such an analysis since we believe it will help address the question of how phenomenologically relevant the BFKL dynamics is at present energies. It would also serve as a very good test of models describing multiple interactions and to gauge how important those effects can be.

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References

- [1] L. N. Lipatov, Sov. Phys. JETP **63** (1986) 904 [Zh. Eksp. Teor. Fiz. **90** (1986) 1536].
- [2] I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822 [Yad. Fiz. **28** (1978) 1597].
- [3] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **45** (1977) 199 [Zh. Eksp. Teor. Fiz. **72** (1977) 377].
- [4] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **44** (1976) 443 [Zh. Eksp. Teor. Fiz. **71** (1976) 840] [Erratum-ibid. **45** (1977) 199].
- [5] L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338 [Yad. Fiz. **23** (1976) 642].
- [6] V. S. Fadin, E. A. Kuraev and L. N. Lipatov, Phys. Lett. B **60** (1975) 50.
- [7] V. S. Fadin and L. N. Lipatov, Phys. Lett. B **429** (1998) 127 [hep-ph/9802290].
- [8] M. Ciafaloni and G. Camici, Phys. Lett. B **430** (1998) 349 [hep-ph/9803389].

- [9] A. H. Mueller and H. Navelet, Nucl. Phys. B **282** (1987) 727.
- [10] D. Y. Ivanov and A. Papa, JHEP **1207**, 045 (2012) doi:10.1007/JHEP07(2012)045 [arXiv:1205.6068 [hep-ph]].
- [11] F. G. Celiberto, D. Y. Ivanov, B. Murdaca and A. Papa, arXiv:1604.08013 [hep-ph].
- [12] V. Del Duca and C. R. Schmidt, Phys. Rev. D **49** (1994) 4510 [hep-ph/9311290].
- [13] W. J. Stirling, Nucl. Phys. B **423** (1994) 56 [hep-ph/9401266].
- [14] L. H. Orr and W. J. Stirling, Phys. Rev. D **56** (1997) 5875 [hep-ph/9706529].
- [15] J. Kwiecinski, A. D. Martin, L. Motyka and J. Outhwaite, Phys. Lett. B **514** (2001) 355 [hep-ph/0105039].
- [16] M. Angioni, G. Chachamis, J. D. Madrigal and A. Sabio Vera, Phys. Rev. Lett. **107**, 191601 (2011) doi:10.1103/PhysRevLett.107.191601 [arXiv:1106.6172 [hep-th]].
- [17] F. Caporale, B. Murdaca, A. Sabio Vera and C. Salas, Nucl. Phys. B **875** (2013) 134 [arXiv:1305.4620 [hep-ph]].
- [18] F. Caporale, D. Y. Ivanov, B. Murdaca and A. Papa, Nucl. Phys. B **877** (2013) 73 [arXiv:1211.7225 [hep-ph]].
- [19] C. Marquet and C. Royon, Phys. Rev. D **79**, 034028 (2009) doi:10.1103/PhysRevD.79.034028 [arXiv:0704.3409 [hep-ph]].
- [20] D. Colferai, F. Schwennsen, L. Szymanowski and S. Wallon, JHEP **1012**, 026 (2010) doi:10.1007/JHEP12(2010)026 [arXiv:1002.1365 [hep-ph]].
- [21] B. Ducloue, L. Szymanowski and S. Wallon, JHEP **1305**, 096 (2013) doi:10.1007/JHEP05(2013)096 [arXiv:1302.7012 [hep-ph]].
- [22] B. Ducloue, L. Szymanowski and S. Wallon, Phys. Lett. B **738**, 311 (2014) doi:10.1016/j.physletb.2014.09.025 [arXiv:1407.6593 [hep-ph]].
- [23] A. H. Mueller, L. Szymanowski, S. Wallon, B. W. Xiao and F. Yuan, JHEP **1603**, 096 (2016) doi:10.1007/JHEP03(2016)096 [arXiv:1512.07127 [hep-ph]].
- [24] A. Sabio Vera, Nucl. Phys. B **746** (2006) 1 [hep-ph/0602250].
- [25] A. Sabio Vera and F. Schwennsen, Nucl. Phys. B **776** (2007) 170 [hep-ph/0702158 [HEP-PH]].
- [26] M. Hentschinski, A. Sabio Vera and C. Salas, Phys. Rev. Lett. **110** (2013) 041601 [arXiv:1209.1353 [hep-ph]].
- [27] M. Hentschinski, A. Sabio Vera and C. Salas, Phys. Rev. D **87** (2013) 076005 [arXiv:1301.5283 [hep-ph]].

- [28] B. Ducloue, L. Szymanowski and S. Wallon, Phys. Rev. Lett. **112** (2014) 082003 [arXiv:1309.3229 [hep-ph]].
- [29] F. Caporale, D. Y. Ivanov, B. Murdaca and A. Papa, Eur. Phys. J. C **74** (2014) 3084 [arXiv:1407.8431 [hep-ph]].
- [30] F. Caporale, D. Y. Ivanov, B. Murdaca and A. Papa, Phys. Rev. D **91** (2015) 11, 114009 [arXiv:1504.06471 [hep-ph]].
- [31] F. G. Celiberto, D. Yu. Ivanov, B. Murdaca and A. Papa, Eur. Phys. J. C **75** (2015) 292 [arXiv:1504.08233 [hep-ph]].
- [32] F. G. Celiberto, D. Y. Ivanov, B. Murdaca and A. Papa, Eur. Phys. J. C **76** (2016) no.4, 224 doi:10.1140/epjc/s10052-016-4053-5 [arXiv:1601.07847 [hep-ph]].
- [33] F. Caporale, G. Chachamis, B. Murdaca and A. Sabio Vera, Phys. Rev. Lett. **116**, no. 1, 012001 (2016) doi:10.1103/PhysRevLett.116.012001 [arXiv:1508.07711 [hep-ph]].
- [34] F. Caporale, F. G. Celiberto, G. Chachamis and A. S. Vera, Eur. Phys. J. C **76**, no. 3, 165 (2016) doi:10.1140/epjc/s10052-016-3963-6 [arXiv:1512.03364 [hep-ph]].
- [35] F. Caporale, F. G. Celiberto, G. Chachamis, D. G. Gomez and A. S. Vera, arXiv:1606.00574 [hep-ph].
- [36] H. Jung, M. Kraemer, A. V. Lipatov and N. P. Zotov, Phys. Rev. D **85**, 034035 (2012) doi:10.1103/PhysRevD.85.034035 [arXiv:1111.1942 [hep-ph]].
- [37] S. P. Baranov, A. V. Lipatov, M. A. Malyshev, A. M. Snigirev and N. P. Zotov, Phys. Lett. B **746**, 100 (2015) doi:10.1016/j.physletb.2015.04.059 [arXiv:1503.06080 [hep-ph]].
- [38] R. Maciula and A. Szczurek, Phys. Lett. B **749**, 57 (2015) doi:10.1016/j.physletb.2015.07.035 [arXiv:1503.08022 [hep-ph]].
- [39] R. Maciula and A. Szczurek, Phys. Rev. D **90**, no. 1, 014022 (2014) doi:10.1103/PhysRevD.90.014022 [arXiv:1403.2595 [hep-ph]].
- [40] K. Kutak, R. Maciula, M. Serino, A. Szczurek and A. van Hameren, arXiv:1602.06814 [hep-ph].
- [41] B. Ducloue, L. Szymanowski and S. Wallon, Phys. Rev. D **92**, no. 7, 076002 (2015) doi:10.1103/PhysRevD.92.076002 [arXiv:1507.04735 [hep-ph]].
- [42] F. Caporale, D. Yu. Ivanov, B. Murdaca, A. Papa, A. Perri, JHEP **1202** (2012) 101; [arXiv:1212.0487 [hep-ph]].
- [43] V.S. Fadin, R. Fiore, M.I. Kotsky and A. Papa, Phys. Lett. D **61** (2000) 094005 [arXiv:9908264 [hep-ph]].

- [44] V.S. Fadin, R. Fiore, M.I. Kotsky and A. Papa, Phys. Lett. D **61** (2000) 094006 [arXiv:9908265 [hep-ph]].
- [45] M. Ciafaloni, Phys. Lett. B **429**, 363 (1998) doi:10.1016/S0370-2693(98)00249-4 [hep-ph/9801322].
- [46] M. Ciafaloni and D. Colferai, Nucl. Phys. B **538**, 187 (1999) doi:10.1016/S0550-3213(98)00621-X [hep-ph/9806350].
- [47] J. Bartels, D. Colferai and G. P. Vacca, Eur. Phys. J. C **24**, 83 (2002) doi:10.1007/s100520200919 [hep-ph/0112283].
- [48] J. Bartels, D. Colferai and G. P. Vacca, Eur. Phys. J. C **29**, 235 (2003) doi:10.1140/epjc/s2003-01169-5 [hep-ph/0206290].
- [49] A.D. Martin, W.J. Stirling, R.S. Thorne and G. Watt, Eur. Phys. J. C **63** (2009) 189 [arXiv:0901.0002 [hep-ph]].
- [50] G.P. Lepage, J. Comput. Phys. **27** (1978) 192.
- [51] T. Hahn, Comput. Phys. Commun. **168** (2005) 78 [arXiv:1408.6373 [hep-ph]].
- [52] T. Hahn, J. Phys. Conf. Ser. **608** (2015) 1 [arXiv:hep-ph/0404043].
- [53] R. Piessens, E. De Doncker-Kapenga and C. W. berhuber, Springer, ISBN: 3-540-12553-1, 1983.
- [54] W. J. Cody, A. J. Strecok and H. C. Thacher, Math. Comput. **27** (1973) 121.
- [55] G. Chachamis, M. Deak, A. Sabio Vera and P. Stephens, Nucl. Phys. B **849** (2011) 28 [arXiv:1102.1890 [hep-ph]].
- [56] G. Chachamis and A. Sabio Vera, Phys. Lett. B **709** (2012) 301 [arXiv:1112.4162 [hep-th]].
- [57] G. Chachamis and A. Sabio Vera, Phys. Lett. B **717** (2012) 458 [arXiv:1206.3140 [hep-th]].
- [58] G. Chachamis, A. Sabio Vera and C. Salas, Phys. Rev. D **87** (2013) 1, 016007 [arXiv:1211.6332 [hep-ph]].
- [59] F. Caporale, G. Chachamis, J. D. Madrigal, B. Murdaca and A. Sabio Vera, Phys. Lett. B **724** (2013) 127 [arXiv:1305.1474 [hep-th]].
- [60] G. Chachamis and A. Sabio Vera, arXiv:1511.03548 [hep-ph].
- [61] G. Chachamis and A. Sabio Vera, JHEP **1602** (2016) 064 doi:10.1007/JHEP02(2016)064 [arXiv:1512.03603 [hep-ph]].